On a property of branching coefficients for affine Lie algebras

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Abstract

It is demonstrated that decompositions of integrable highest weight modules of a simple Lie algebra with respect to its reductive subalgebra obey the set of algebraic relations leading to the recursive properties for the corresponding branching coefficients. These properties are encoded in the special element $\Gamma_{\mathfrak{g}\supset\mathfrak{a}}$ of the formal algebra $\mathcal{E}_{\mathfrak{a}}$ that describes the injection and is called the fan. In the simplest case, when $\mathfrak{a}=\mathfrak{h}(\mathfrak{g})$, the recursion procedure generates the weight diagram of a module $L_{\mathfrak{g}}$. When applied to a reduction of highest weight modules the recursion described by the fan provides a highly effective tool to obtain the explicit values of branching coefficients.

1 Introduction

We consider integrable modules L^{μ} of affine Lie algebra $\mathfrak{g}$ with the highest weight μ and the reduced modules $L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}}$ with respect to a reductive subalgebra $\mathfrak{a}\subset\mathfrak{g}$. In particular when the Cartan subalgebra $\mathfrak{a}=\mathfrak{h}$ is studied, the branching

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coefficients indicate the dimensions of the weight subspaces and thus describe the module. String functions and branching coefficients of the affine Lie algebra pairs (e.g. $A_n^{(1)} \subset A_{n-p-1}^{(1)} \oplus A_p^{(1)}$) arise in the computation of the local state probabilities for solvable models on square lattice [1]. Irreducible highest weight modules with dominant integral weights appear also in application of the quantum inverse scattering method [2] where solvable spin chains are studied in the framework of the AdS/CFT correspondence conjecture of the super-string theory (see [3, 4] and references therein).

There are different ways to find branching coefficients. One can use the BGG resolution [7] (for Kac-Moody algebras the algorithm is described in [5, 6]), the Schure function series [8], the BRST cohomology [9], Kac-Peterson formulas [5] or the combinatorial methods applied in [10]. We want to obtain the recursive formulas for weight multiplicities and branching coefficients using the purely algebraic approach. From the Weyl-Kac character formula [5]

$$\mathrm{ch}L^{\mu}\left(\mathfrak{g}\right) = \frac{\sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}, \tag{1}$$

we derive the special set of relations for branching coefficients. These relations can be used both to construct a representation and to reduce it with respect to a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ that is to find the corresponding branching rules. Each relation of the set deals with a finite collection of weights. Among them it is always possible to fix the lowest one (with respect to the natural ordering of weights induced by basic roots). Thus it is possible to use the relations of the set as recurrent relations for branching coefficients. It is demonstrated that branching is governed by a certain system of weights (called "the fan of injection") that depends only on the algebra and the injection morphism and can be used to decompose the highest weight modules.

For finite dimensional classical Lie algebras the case of regular injections was considered in [11] where the recurrent relations were constructed using the properties of the Kostant-Heckman partition function. The same method was used for regular injections of affine Lie algebras [12]. In this study we present a different approach and find that for any reductive subalgebra $\mathfrak{a}$ of an affine Lie algebra $\mathfrak{g}$ such that $\mathfrak{h}_{\mathfrak{a}}^{*} \subset \mathfrak{h}_{\mathfrak{g}}^{*}$ and $\mathfrak{h}_{\overset{\circ}{\mathfrak{a}}}^{*} \subset \mathfrak{h}_{\overset{\circ}{\mathfrak{g}}}^{*}$ the branching coefficients obey the set of properties that give rise to recurrent relations. The latter provide a compact and effective method to construct the corresponding branching rules. The results are illustrated by examples.

2 Basic definitions and relations.

Consider the affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with the underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}_{\mathfrak{a}}^{*} \subset \mathfrak{h}_{\mathfrak{g}}^{*}$ and $\mathfrak{h}_{\overset{\circ}{\mathfrak{a}}}^{*} \subset \mathfrak{h}_{\overset{\circ}{\mathfrak{g}}}^{*}$.

The following notation will also be used:

L^{μ} $(L^{\nu}_{\mathfrak{a}})$ – the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$-module with the highest weight ν);

r , $(r_{\mathfrak{a}})$ – the rank of the algebra $\mathfrak{g}$ (resp. $\mathfrak{a}$) ;

Δ $(\Delta_{\mathfrak{a}})$– the root system; Δ^{+} (resp. $\Delta^{+}_{\mathfrak{a}}$)– the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\,(\alpha)$ $(\mathrm{mult}_{\mathfrak{a}}\,(\alpha))$ – the multiplicity of the root α in Δ (resp. in $(\Delta_{\mathfrak{a}})$);

$\overset{\circ}{\Delta}$, $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ – the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$);

$\mathcal{N}^{\mu}$, $(\mathcal{N}^{\nu}_{\mathfrak{a}})$ – the weight diagram of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$) ;

W , $(W_{\mathfrak{a}})$– the corresponding Weyl group;

C , $(C_{\mathfrak{a}})$– the fundamental Weyl chamber;

ρ , $(\rho_{\mathfrak{a}})$ – the Weyl vector;

$\epsilon\,(w) := \det\,(w)$;

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ – the i-th (resp. j-th) basic root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}})$;

δ – the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha^{\vee}_i$, $\left(\alpha^{\vee}_{(\mathfrak{a})j}\right)$– the basic coroot for $\mathfrak{g}$ (resp. $\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\overset{\circ}{\xi}$, $\overset{\circ}{\xi}_{(\mathfrak{a})}$ – the finite (classical) part of the weight $\xi \in P$, $\left(\text{resp. } \xi_{(\mathfrak{a})} \in P_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ – the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level k and grade n .

P (resp. $P_{\mathfrak{a}}$) – the weight lattice;

M (resp. $M_{\mathfrak{a}}$) :=

$$= \left\{ \begin{array}{c} \sum_{i=1}^{r} \mathbf{Z}\alpha^{\vee}_i \ \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha^{\vee}_{(\mathfrak{a})i}\right) \text{for untwisted algebras or } A^{(2)}_{2r}, \\ \sum_{i=1}^{r} \mathbf{Z}\alpha_i \ \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}\right) \text{for } A^{(u\geq 2)}_{r} \text{ and } A \neq A^{(2)}_{2r}, \end{array} \right\};$$

$\mathcal{E}$, $(\mathcal{E}_{\mathfrak{a}})$– the group algebra of the group P (resp. $P_{\mathfrak{a}}$);

$\Theta_{\lambda} := e^{-\frac{|\lambda|^2}{2k}\delta} \sum\limits_{\alpha \in M} e^{t_{\alpha} \circ \lambda}$ – the classical theta-function;

$\Theta_{(\mathfrak{a})\nu} := e^{-\frac{|\nu|^2}{2k_{\mathfrak{a}}}\delta} \sum\limits_{\beta \in M_{\mathfrak{a}}} e^{t_{\beta} \circ \nu}$;

notice that when the injection is considered the level $k_{\mathfrak{a}}$ must be correlated with the corresponding rescaling of roots;

$$A_{\lambda} := \sum_{s \in \overset{\circ}{W}} \epsilon(s)\Theta_{s\circ\lambda} \left(\text{resp. } A_{(\mathfrak{a})\nu} := \sum_{s \in \overset{\circ}{W}_{\mathfrak{a}}} \epsilon(s)\Theta_{(\mathfrak{a})s\circ\nu}\right);$$

$$\Psi^{(\mu)} := e^{\frac{|\mu+\rho|^2}{2k}\delta \, - \, \rho} A_{\mu+\rho} = e^{\frac{|\mu+\rho|^2}{2k}\delta \, - \, \rho} \sum_{s \in \overset{\circ}{W}} \epsilon(s)\Theta_{s\circ(\mu+\rho)} =$$

$$= \sum_{w \in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}$$ – the singular weight element for the $\mathfrak{g}$-module L^{μ};

$$\Psi^{(\nu)}_{(\mathfrak{a})} := e^{\frac{|\nu+\rho_{\mathfrak{a}}|^2}{2k_{\mathfrak{a}}}\delta \, - \, \rho_{\mathfrak{a}}} A_{(\mathfrak{a})\nu+\rho_{\mathfrak{a}}} = e^{\frac{|\nu+\rho_{\mathfrak{a}}|^2}{2k_{\mathfrak{a}}}\delta \, - \, \rho_{\mathfrak{a}}} \sum_{s \in \overset{\circ}{W}_{\mathfrak{a}}} \epsilon(s)\Theta_{(\mathfrak{a})s\circ\left(\nu+\rho_{\mathfrak{a}}\right)} =$$

$= \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w \circ (\nu + \rho_{\mathfrak{a}}) - \rho_{\mathfrak{a}}}$ – the corresponding singular weight element for the $\mathfrak{a}$-module $L_{\mathfrak{a}}^{\nu}$;

$\widehat{\Psi^{(\mu)}}$ $\left(\widehat{\Psi_{(\mathfrak{a})}^{(\nu)}}\right)$ – the set of singular weights $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) for the module L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$) with the coordinates $\left(\overset{\circ}{\xi}, k, n, \epsilon\left(w\left(\xi\right)\right)\right) |_{\xi = w(\xi) \circ (\mu + \rho) - \rho}$, (resp. $\left(\overset{\circ}{\xi}, k, n, \epsilon\left(w_a\left(\xi\right)\right)\right) |_{\xi = w_a(\xi) \circ (\nu + \rho_a) - \rho_a}$), (this set is similar to $P'_{\text{nice}}(\mu)$ in [6])

$m_{\xi}^{(\mu)}$, $\left(m_{\xi}^{(\nu)}\right)$ – the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right)$ (resp. $ch\left(L_{\mathfrak{a}}^{\nu}\right)$)– the formal character of L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right) = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu + \rho) - \rho}}{\prod_{\alpha \in \Delta^{+}} (1 - e^{-\alpha})^{\text{mult}(\alpha)}} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$ – the Weyl-Kac formula.

$R := \prod_{\alpha \in \Delta^{+}} \left(1 - e^{-\alpha}\right)^{\text{mult}(\alpha)} = \Psi^{(0)}$
$\left(\text{resp. } R_{\mathfrak{a}} := \prod_{\alpha \in \Delta_{\mathfrak{a}}^{+}} \left(1 - e^{-\alpha}\right)^{\text{mult}_{\mathfrak{a}}(\alpha)} = \Psi_{\mathfrak{a}}^{(0)}\right)$– the denominator.

3 Anomalous multiplicities and recurrent relations

For the injection $\mathfrak{a} \longrightarrow \mathfrak{g}$ consider the reduced module

$$L_{\mathfrak{g} \downarrow \mathfrak{a}}^{\mu} = \bigoplus_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} L_{\mathfrak{a}}^{\nu} \tag{2}$$

with the branching coefficients $b_{\nu}^{(\mu)}$. The character reduction

$$\pi_{\mathfrak{a}} \circ \left(\text{ch} L_{\mathfrak{g}}^{\mu}\right) = \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \text{ch} L_{\mathfrak{a}}^{\nu} \tag{3}$$

involves the projection operator $\pi_{\mathfrak{a}} : P \longrightarrow P_{\mathfrak{a}}$.

The denominator identity can be applied to redress the relation (3),

$$\frac{\pi_{\mathfrak{a}} \circ \left(\sum_{w \in W} \epsilon(w) e^{w \circ (\mu + \rho) - \rho}\right)}{\pi_{\mathfrak{a}} \circ \left(\prod_{\alpha \in \Delta^{+}} \left(1 - e^{-\alpha}\right)^{\text{mult}(\alpha)}\right)} = \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \frac{\sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w \circ (\nu + \rho_{\mathfrak{a}}) - \rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{a}}^{+}} \left(1 - e^{-\beta}\right)^{\text{mult}_{\mathfrak{a}}(\beta)}}, \tag{4}$$

and to rewrite it as

$$\begin{aligned} &\pi_{\mathfrak{a}} \circ \left(\sum_{w \in W} \epsilon(w) e^{w \circ (\mu + \rho) - \rho}\right) = \\ &= \frac{\pi_{\mathfrak{a}} \circ \left(\prod_{\alpha \in \Delta^{+}} \left(1 - e^{-\alpha}\right)^{\text{mult}(\alpha)}\right)}{\prod_{\beta \in \Delta_{\mathfrak{a}}^{+}} (1 - e^{-\beta})^{\text{mult}_{\mathfrak{a}}(\beta)}} \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w \circ (\nu + \rho_{\mathfrak{a}}) - \rho_{\mathfrak{a}}}, \end{aligned} \tag{5}$$

For the trivial $\mathfrak{g}$-module L^0 with $\mu = 0$ we have

$$\frac{\pi_{\mathfrak{a}} \circ \left(\sum_{w\in W} \epsilon(w) e^{w\circ\rho-\rho}\right)}{\pi_{\mathfrak{a}} \circ \left(\prod_{\alpha\in\Delta^+} \left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}\right)} = \frac{\sum\limits_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ\rho_{\mathfrak{a}}-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta^+_{\mathfrak{a}}} \left(1-e^{-\beta}\right)^{\mathrm{mult}_{\mathfrak{a}}(\beta)}},$$

$$\frac{\pi_{\mathfrak{a}} \circ \left(\prod_{\alpha\in\Delta^+} \left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}\right)}{\prod_{\beta\in\Delta^+_{\mathfrak{a}}} \left(1-e^{-\beta}\right)^{\mathrm{mult}_{\mathfrak{a}}(\beta)}} = \frac{\pi_{\mathfrak{a}} \circ \left(\sum_{w\in W} \epsilon(w) e^{w\circ\rho-\rho}\right)}{\sum\limits_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ\rho_{\mathfrak{a}}-\rho_{\mathfrak{a}}}}.$$

The relation (5) takes the form

$$\begin{gathered}\pi_{\mathfrak{a}} \circ \left(\sum_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}\right) \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ\rho_{\mathfrak{a}}-\rho_{\mathfrak{a}}} = \\ = \pi_{\mathfrak{a}} \circ \left(\sum_{w\in W} \epsilon(w) e^{w\circ\rho-\rho}\right) \sum_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu} \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}.\end{gathered} \tag{6}$$

Consider the expression $\sum_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu} \Psi^{(\nu)}_{(\mathfrak{a})}$ and introduce the numbers $k^{(\mu)}_{\lambda}$, called *the anomalous branching coefficients*, – the multiplicities of submodules $L^{\nu}_{\mathfrak{a}}$ times the determinants $\epsilon(w)$ contained in $\Psi^{(\nu)}_{(\mathfrak{a})}$.

$$\sum_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu} \Psi^{(\nu)}_{(\mathfrak{a})} = \sum_{\lambda\in P_{\mathfrak{a}}} k^{(\mu)}_{\lambda} e^{\lambda} \tag{7}$$

In these terms the expression (6) reads

$$\begin{gathered}\pi_{\mathfrak{a}} \circ \left(\sum_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}\right) \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ\rho_{\mathfrak{a}}-\rho_{\mathfrak{a}}} = \\ = \pi_{\mathfrak{a}} \circ \left(\sum_{w\in W} \epsilon(w) e^{w\circ\rho-\rho}\right) \sum_{\xi\in P_{\mathfrak{a}}} k^{(\mu)}_{\xi} e^{\xi},\end{gathered} \tag{8}$$

or

$$\begin{gathered}\sum_{w\in W, v\in W_{\mathfrak{a}}} \epsilon(w)\epsilon(v) e^{\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho)+v\circ\rho_{\mathfrak{a}}-\rho_{\mathfrak{a}}} = \\ = \sum_{\xi\in P_{\mathfrak{a}}} \sum_{w\in W} \epsilon(w) e^{\pi_{\mathfrak{a}}\circ(w\circ\rho-\rho)+\xi} k^{(\mu)}_{\xi}.\end{gathered} \tag{9}$$

Thus we have proved the statement:

Proposition 1 *Let L^{μ} be the integrable highest weight module of $\mathfrak{g}$, $\mathfrak{a} \subset \mathfrak{g}$, $\mathfrak{h}_{\mathfrak{a}} \subset \mathfrak{h}_{\mathfrak{g}}$, $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$ and $\pi_{\mathfrak{a}}$ – a projection $P \longrightarrow P_{\mathfrak{a}}$ – then for any point $\xi \in P_{\mathfrak{a}}$ the following relation holds:*

$$\sum_{w\in W} \epsilon(w) k^{(\mu)}_{\xi-\pi_{\mathfrak{a}}\circ(w\circ\rho-\rho)} = \sum_{w\in W, v\in W_{\mathfrak{a}}} \epsilon(w)\epsilon(v) \delta_{\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho),\xi+\rho_{\mathfrak{a}}-v\circ\rho_{\mathfrak{a}}}. \tag{10}$$

This can be rearranged to produce a recurrent relation for the anomalous multiplicities,

$$k_{\xi}^{(\mu)} = -\sum_{w\in W\setminus e} \epsilon(w) k^{(\mu)}_{\xi-\pi_{\mathfrak{a}}\circ(w\circ\rho-\rho)} + \sum_{w\in W, v\in W_{\mathfrak{a}}} \epsilon(w)\epsilon(v)\delta_{\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho),\xi+\rho_{\mathfrak{a}}-v\circ\rho_{\mathfrak{a}}}. \tag{11}$$

This formula can be applied to find the branching coefficients $b_{\nu}^{(\mu)}$ due to the fact that being restricted to the fundamental Weyl chamber ($C_{\mathfrak{a}}$) the anomalous branching coefficients coincide with the branching coefficients

$$k_{\xi}^{(\mu)} = b_{\xi}^{(\mu)} \qquad \text{for} \quad \xi \in C_{\mathfrak{a}}.$$

The relation (11) contains the standard system of shifts, $\xi \longrightarrow \xi - \pi_{\mathfrak{a}} \circ (w\circ\rho-\rho)$, defined by the singular weights of the trivial module, the corresponding element of the algebra $\mathcal{E}$ being $\Psi^{(0)} = e^{\frac{|\rho|^2}{2k}\delta\ -\ \rho} \times \sum_{s\in \overset{\circ}{W}} \epsilon(s)\Theta_{s\circ(\rho)}$. At the same time the second term in the r.h.s. contains the summation in both W and $W_{\mathfrak{a}}$. Below we demonstrate that this relation can be simplified by introducing the different system of shifts.

Let us return to the relation (5). The conditions $\mathfrak{a} \longrightarrow \mathfrak{g}$ and $\mathfrak{h}_{\mathfrak{a}} \subset \mathfrak{h}_{\mathfrak{g}}$ guarantee the inclusion $\Delta_{\mathfrak{a}}^{+} \subset \Delta^{+}$. Thus the first factor in the r. h. s. being an element of $\mathcal{E}$ can be written as

$$\begin{aligned}\frac{\pi_{\mathfrak{a}}\circ\left(\prod_{\alpha\in\Delta^{+}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}\right)}{\prod_{\beta\in\Delta_{\mathfrak{a}}^{+}}\left(1-e^{-\beta}\right)^{\mathrm{mult}_{\mathfrak{a}}(\beta)}} &= \prod_{\alpha\in(\pi_{\mathfrak{a}}\circ\Delta^{+})}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)} = \\ &= -\sum_{\gamma\in P_{\mathfrak{a}}} s(\gamma)\, e^{-\gamma}.\end{aligned}$$

For the coefficient function $s(\gamma)$ define $\Phi_{\mathfrak{a}\subset\mathfrak{g}} \subset P_{\mathfrak{a}}$ as its carrier:

$$\Phi_{\mathfrak{a}\subset\mathfrak{g}} = \{\gamma \in P_{\mathfrak{a}} \mid s(\gamma) \neq 0\}; \tag{12}$$

$$\prod_{\alpha\in(\pi_{\mathfrak{a}}\circ\Delta^{+})}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)} = -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma)\, e^{-\gamma}. \tag{13}$$

When the second factor in the r.h.s. of (5) is also decomposed we obtain the relation

$$\begin{aligned}\sum_{w\in W}\epsilon(w)e^{\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho)} &= -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma)\, e^{-\gamma} \sum_{\lambda\in P_{\mathfrak{a}}} k_{\lambda}^{(\mu)} e^{\lambda} \\ &= -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}} s(\gamma)\, k_{\lambda}^{(\mu)} e^{\lambda-\gamma}\end{aligned}$$

and the new property

$$\sum_{w\in W}\epsilon(w)\,\delta_{\xi,\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho)} + \sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma)\, k_{\xi+\gamma}^{(\mu)} = 0; \qquad \xi \in P_{\mathfrak{a}}. \tag{14}$$

Thus the following statement is true:

Proposition 2 *Let L^{μ} be the integrable highest weight module of $\mathfrak{g}$, $\mathfrak{a} \subset \mathfrak{g}$, $\mathfrak{h}_{\mathfrak{a}} \subset \mathfrak{h}_{\mathfrak{g}}$, $\mathfrak{h}_{\mathfrak{a}}^{*} \subset \mathfrak{h}_{\mathfrak{g}}^{*}$ and $\pi_{\mathfrak{a}}$ – a projection $P \longrightarrow P_{\mathfrak{a}}$ then for any vector $\xi \in P_{\mathfrak{a}}$ the sum $\sum_{\gamma \in \Phi_{\mathfrak{a} \subset \mathfrak{g}}} -s(\gamma) k_{\xi+\gamma}^{(\mu)}$ is equal to the anomalous multiplicity of the weight ξ in the module $\pi_{\mathfrak{a}} \circ L_{\mathfrak{g}}^{\mu}$.*

This property (14) also produces recurrent relations for the anomalous multiplicities. Returning to the relation (13) we see that $\Phi_{\mathfrak{a} \subset \mathfrak{g}}$ contains vectors with nonnegative grade and is a subset in the carrier of the singular weights element $\Psi^{(\mu)} = e^{\frac{|\mu+\rho|^2}{2k}\delta - \rho} \sum_{s \in \overset{\circ}{W}} \epsilon(s) \Theta_{s \circ (\mu+\rho)}$. In each grade the set $\Phi_{\mathfrak{a} \subset \mathfrak{g}}$ has finite number of vectors [5]. In particular

$$\# \left(\Phi_{\mathfrak{a} \subset \mathfrak{g}}\right)_{n=0} = \# \overset{\circ}{W}.$$

In $\left(\Phi_{\mathfrak{a} \subset \mathfrak{g}}\right)_{n=0}$ let γ_0 be the lowest vector with respect to the natural ordering in $\overset{\circ}{\Delta}_{\mathfrak{a}}$. Decomposing the defining relation (13),

$$\prod_{\alpha \in (\pi_{\mathfrak{a}} \circ \Delta^{+})} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}(\alpha) - \mathrm{mult}_{\mathfrak{a}}(\alpha)} = -s\left(\gamma_0\right) e^{-\gamma_0} - \sum_{\gamma \in \Phi_{\mathfrak{a} \subset \mathfrak{g}} \setminus \gamma_0} s(\gamma) e^{-\gamma}, \quad (15)$$

in (14) we obtain

$$k_{\xi}^{(\mu)} = -\frac{1}{s\left(\gamma_0\right)} \left(\sum_{w \in W} \epsilon(w) \delta_{\xi, \pi_{\mathfrak{a}} \circ (w \circ (\mu+\rho) - \rho) + \gamma_0} + \sum_{\gamma \in \Gamma_{\mathfrak{a} \subset \mathfrak{g}}} s\left(\gamma + \gamma_0\right) k_{\xi+\gamma}^{(\mu)} \right) \quad (16)$$

where the set

$$\Gamma_{\mathfrak{a} \subset \mathfrak{g}} = \left\{ \xi - \gamma_0 | \xi \in \Phi_{\mathfrak{a} \subset \mathfrak{g}} \right\} \setminus \{0\}. \quad (17)$$

was introduced called *the fan of the injection* $\mathfrak{a} \subset \mathfrak{g}$. The equality (16) can be considered as a recurrent relation for anomalous branching coefficients $k_{\xi}^{(\mu)}$. Contrary to the relation (11) here only the summation over W is applied.

When $r = r_{\mathfrak{a}}$ the positive roots $\Delta_{\mathfrak{a}}^{+}$ can always be chosen so that $\gamma_0 = 0$, the relation (15) indicates that $s\left(\gamma_0\right) = -1$. Thus in this special case

$$\Gamma_{\mathfrak{a} \subset \mathfrak{g}} = \Phi_{\mathfrak{a} \subset \mathfrak{g}} \setminus \{0\},$$

and the recurrent relation acquires the form

$$k_{\xi}^{(\mu)} = \sum_{\gamma \in \Gamma_{\mathfrak{a} \subset \mathfrak{g}}} s(\gamma) k_{\xi+\gamma}^{(\mu)} + \sum_{w \in W} \epsilon(w) \delta_{\xi, \pi_{\mathfrak{a}} \circ (w \circ (\mu+\rho) - \rho)}. \quad (18)$$

Comments:

1. The sets $\Phi_{\mathfrak{a} \subset \mathfrak{g}}$ and $\Gamma_{\mathfrak{a} \subset \mathfrak{g}}$ do not depend on the representation $L^{\mu}(\mathfrak{g})$ and describe the injection of the subalgebra $\mathfrak{a}$ into the algebra $\mathfrak{g}$.

2. Let the set of singular weights of the projected module $\pi_{\mathfrak{a}} \circ L^{\mu}$ be constructed. Then the sets $\Psi^{(0)}$ and $\Psi^{(0)}_{(\mathfrak{a})}$ define the anomalous branching coefficients for the reduced module $L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}}$ by means of relation (11). The same information can be obtained using the fan $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ via the relations (16) or (18).

3. The set of branching coefficients $\left\{b_{\nu}^{(\mu)}\right\}$ is the subset of the anomalous branching coefficients $\left\{k_{\xi}^{(\mu)}\right\}$:

$$\left\{b_{\nu}^{(\mu)} \mid \nu \in P_{\mathfrak{a}}^{+}\right\} = \left\{k_{\xi}^{(\mu)} \mid \xi \in \overline{C_{\mathfrak{a}}}\right\}.$$

   Thus the recurrent relations (11),(16) and (18) supply us with the branching coefficients as well.

Let us apply the obtained results to the case where $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{a} = \mathfrak{h}_{\mathfrak{g}}$. Then the Weyl group $W_{\mathfrak{a}}$ and the projector $\pi_{\mathfrak{a}}$ are trivial and in the formulas (11) and (18) the anomalous coefficient $k_{\xi}^{(\mu)}$ is the multiplicity of the only singular weight that contributes to the element $\Psi^{(\xi)}_{\mathfrak{h}_{\mathfrak{g}}}$ – the highest weight of the $\mathfrak{h}_{\mathfrak{g}}$-submodule. This means that here the number $k_{\xi}^{(\mu)}$ is always nonnegative and coincides with the multiplicity $m_{\xi}^{(\mu)}$ of the weight ξ in the module $L^{\mu}(\mathfrak{g})$. The relations (11) and (18) directly lead to

Corollary 3 *In the integrable highest weight module $L^{\mu}(\mathfrak{g})$ of an affine Lie algebra $\mathfrak{g}$ the multiplicity $m_{\xi}^{(\mu)}$ of the weight ξ (considered as a numerical function on $P_{\mathfrak{g}}$) obeys the relation*

$$m_{\xi}^{(\mu)} = -\sum_{w \in W \setminus e} \epsilon(w) m^{(\mu)}_{\xi-(w\circ\rho-\rho)} + \sum_{w\in W} \epsilon(w)\delta_{(w\circ(\mu+\rho)-\rho),\xi}. \qquad (19)$$

In implicit form this relation can be found for affine Lie algebras in [5] (Ch.11, the second formula in Ex. 11.14) and for finite dimensional algebras in [13]. Usually the truncated formula (without the second term in the r. h. s.) is presented as the recurrent relation for the multiplicities of weights (see for example [14], Ch. VIII, Sect. 9.3). From our point of view it is highly important to deal with the recurrent relation in its full form. The reason is that the relation (19) can be applied for any reducible module and is valid in any domain of P, not only inside the diagram $\mathcal{N}^{\mu} \setminus \mu$ with the single highest weight μ.

This relation gives the possibility to construct recursively the module $L^{\mu}(\mathfrak{g})$ provided the elements $\Psi^{(\mu)}$ and $\Psi^{(0)}$ are known.

4 Examples

Example 1 *Consider the finite dimensional Lie algebras* $A_2 \subset g_2$. *The root system* Δ *is generated by the simple roots* α_1 *(the long) and* α_2 *(the short) with the angle* $\frac{5\pi}{6}$ *between them.*

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\} \tag{20}$$

$$\Delta^+_{sl(3)} = \{\alpha_1, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\} \tag{21}$$

From (12), (13) and (17) we obtain

$$\Phi_{A_2 \subset g_2} = \{0, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2, 2\alpha_1 + 4\alpha_2\} \tag{22}$$

$$\Gamma_{A_2 \subset g_2} = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2, 2\alpha_1 + 4\alpha_2\} \tag{23}$$

Consider the adjoint module $L^{2\alpha_1+3\alpha_2}$. *Its singular weights are*

$$\begin{gathered}\{2\alpha_1 + 3\alpha_2, 3\alpha_2, -\alpha_1 + 2\alpha_2, -4\alpha_2, -\alpha_1 - 6\alpha_2, -8\alpha_1 - 12\alpha_2, -8\alpha_1 - 13\alpha_2, \\ -6\alpha_1 - 13\alpha_2, -5\alpha_1 - 12\alpha_2, -6\alpha_1 - 6\alpha_2, -5\alpha_1 - 4\alpha_2, 2\alpha_1 + 2\alpha_2\}\end{gathered} \tag{24}$$

to each of them corresponds the Weyl transformation $w(\psi)$ *and the value* $\epsilon(w)$*:*

$$\{\epsilon(w(\psi))\} = \{+1, -1, +1, +1, -1, -1, +1, -1, +1, -1, +1, -1\} \tag{25}$$

In the closure of the fundamental chamber $\overline{C_{\mathfrak{a}}}$ *the relation (18) defines three nonzero branching coefficients*

$$b^{(\mu)}_{2\alpha_1+3\alpha_2} = +1, \quad b^{(\mu)}_{\alpha_1+2\alpha_2} = +1, \quad b^{(\mu)}_{\alpha_1+\alpha_2} = +1. \tag{26}$$

corresponding to the adjoint and two fundamental submodules of $sl(3)$ *in the decomposition* $L^{2\alpha_1+3\alpha_2}_{\downarrow sl(3)}$ *(Notice that we need the singular weight* $2\alpha_1 + 2\alpha_2$ *with* $s(2\alpha_1 + 2\alpha_2) = -1$ *to be used in the above calculations.)*

Example 2 *Consider the special injection of the algebra* B_1 *into* A_2. *Let* α_1 *and* α_2 *be the simple roots of* A_2*,*

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \tag{27}$$

The only positive root of B_1 *is*

$$\Delta^+_{B_1} = \left\{\beta := \frac{1}{2}\alpha_1\right\} \tag{28}$$

From (12) it follows that

$$\Phi_{B_1 \subset A_2} = \left\{0, \alpha_1, -\frac{1}{2}\alpha_1, \frac{1}{2}\alpha_1\right\} = \{0, 2\beta, -\beta, \beta\} \tag{29}$$

For these vectors the function $s_{B_1 \subset A_2}$ *has the values*

$$s_{B_1 \subset A_2} = \{-1, +1, +1, -1\} \tag{30}$$

The minimal vector γ_0

$$\begin{aligned} \gamma_0^{B_1\subset A_2} &= -\beta \\ s_{B_1\subset A_2}(-\beta) &= +1. \end{aligned} \tag{31}$$

The fan is formed by eliminating γ_0 *from* $\Phi_{B_1\subset A_2}$ *and shifting the remaining vectors by* $-\gamma_0$*:*

$$\begin{aligned} \Gamma_{B_1\subset A_2} &= \{\beta, 2\beta, 3\beta\} \\ s_{B_1\subset A_2}(\gamma+\gamma_0) &= \{-,-,+\}; \qquad \gamma\in\Gamma_{B_1\subset A_2} \end{aligned} \tag{32}$$

Consider the module $L^{\alpha_1+\alpha_2}$*. Its singular weights are*

$$\{\alpha_1+\alpha_2, -\alpha_1+\alpha_2, -3\alpha_1-\alpha_2, -3\alpha_1, -\alpha_1-3\alpha_2, \alpha_1-\alpha_2\} \tag{33}$$

with the values

$$\{\epsilon(\xi)\} = \{+1, \quad -1, \quad +1, \quad -1, \quad +1, \quad -1\} \tag{34}$$

We rewrite their projections to P_{B_1} *in terms of* β*:*

$$\begin{aligned} & \{-6\beta, -4\beta, -4\beta, 0, 0, +2\beta\} \\ \{\epsilon(\xi)\} &= \{+1, -1, +1, -1, +1, -1\} \end{aligned} \tag{35}$$

In the closure of the fundamental chamber $\left(\overline{C_{\mathfrak{a}}}\right)$ *the relation (16) defines two nonzero branching coefficients*

$$b_{2\beta}^{(\mu)} = +1, \quad b_{\beta}^{(\mu)} = +1. \tag{36}$$

corresponding to the submodule of the adjoint subrepresentation and the 5-dimensional spin 2 submodule of B_1 *in the reduced module* $L_{\downarrow B_1}^{\alpha_1+\alpha_2}$ *(Notice that the singular vector "0" with* $s(0) = -1$ *has the multiplicity 2.)*

Example 3 *For the affine algebra* $A_2^{(1)}$ *consider the twisted subalgebra* $A_2^{(2)}$*. For the level* k *sublattice* P_k *introduce the normalized basic vectors* $\{e_1, e_2, e_3\}$ *with* $\mid e_j\mid_{j=1,2,3} = 1$ *and* δ *with* $\mid\delta\mid = 0$ *. For* $A_2^{(1)}$ *we fix the simple roots*

$$\alpha_1 = e_1 - e_2; \quad \alpha_2 = e_2 - e_3; \quad \alpha_0 = \delta - e_1 + e_3;$$

The positive roots are as follows:

$$\Delta^+ = \left\{ \begin{array}{c} \alpha_j + l\delta; \quad j = 1,2,3; \quad l\in\mathbf{Z}_{\geq 0} \\ -\alpha_j + p\delta; \quad j = 1,2,3; \quad p\in\mathbf{Z}_{>0} \\ p\delta; \quad mult\,(p\delta) = 2; \quad p\in\mathbf{Z}_{>0} \end{array} \right\},$$

the classical positive roots being

$$\overset{\circ}{\Delta}{}^{+} = \{\alpha_1 = e_1 - e_2; \quad \alpha_2 = e_2 - e_3; \quad \alpha_3 = e_1 - e_3;\}.$$

The fundamental weights

$$\omega_1 = \frac{1}{3}(2e_1 - e_2 - e_3) + k; \quad \omega_2 = \frac{1}{3}(e_2 + e_1 - 2e_3) + k; \quad \omega_0 = k;$$

and the Weyl vector

$$\rho = (\alpha_1 + \alpha_2, 3, 0).$$

The Weyl group is generated by the classical reflections

$$s_{\alpha_1}, s_{\alpha_2}$$

and (in accord with $M = \sum_{i=1}^{r} \mathbf{Z}\alpha_i^\vee$ *for untwisted algebras) the translations*

$$t_{\alpha_1}, t_{\alpha_2}.$$

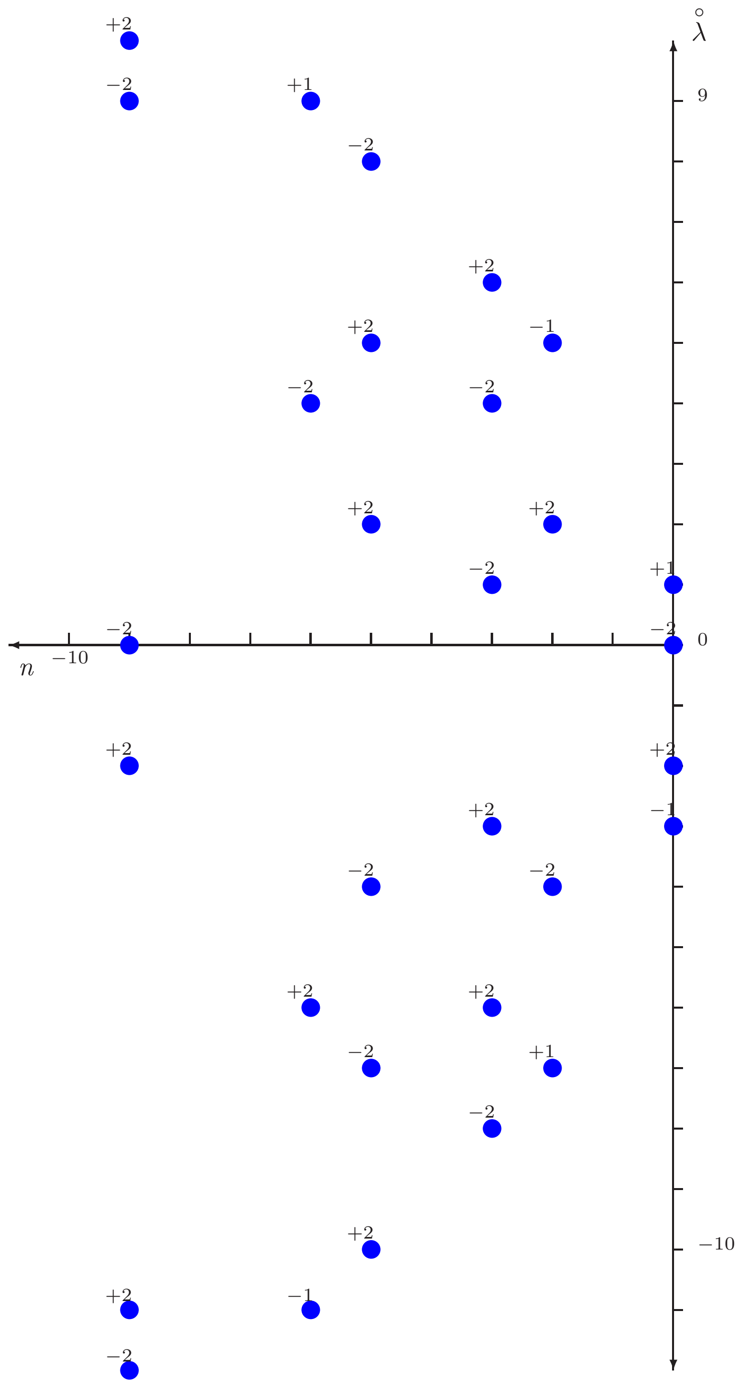
Consider the module L^{ω_0}*. Notice that to obtain the branching rules we need only the projected singular element* $\pi_{\mathfrak{a}} \circ \Psi^{(\omega_0)}$ *of this module and the set* $\Gamma_{A_2^{(2)} \subset A_2^{(1)}}$ *and do not need any other properties of the module itself. Let us describe the element* $\Psi^{(\omega_0)}$ *by the set* $\widehat{\Psi^{(\omega_0)}}$ *of singular weights of the module* L^{ω_0}*:*

$$\left\{ (\lambda_1, \lambda_2, \lambda_3, n, \epsilon(w)) \,|\, \lambda_i \in \mathbf{Z}, n \in \mathbf{Z}_{\leq 0}, \epsilon(w) = \pm 1 \right\},$$

(the level is always $k = 1$ *and is not indicated). Then for* $n > -10$ *the set* $\widehat{\Psi^{(\omega_0)}}$ *contains the following 54 vectors:*

$$\begin{aligned}
\widehat{\Psi^{(\omega_0)}} = \{ & (0,0,0,0,-1), (-1,1,0,0,1), (0,-1,1,0,1), (-1,-1,2,0,-1), \\
& (-2,1,1,0,-1), (-2,0,2,0,1), (2,-3,1,-2,-1), (-4,0,4,-2,-1), \\
& (-4,3,1,-2,1), (-1,-3,4,-2,1), (-1,3,-2,-2,-1), (2,0,-2,-2,1), \\
& (-5,1,4,-3,1), (-4,-1,5,-3,1), (0,-4,4,-3,-1), (-2,-3,5,-3,-1), \\
& (-4,4,0,-3,-1), (-5,3,2,-3,-1), (3,-3,0,-3,1), (2,-4,2,-3,1), \\
& (0,3,-3,-3,1), (-2,4,-2,-3,1), (3,-1,-2,-3,-1), (2,1,-3,-3,-1), \\
& (-5,-1,6,-5,-1), (-6,1,5,-5,-1), (0,-5,5,-5,1), (-2,-4,6,-5,1), \\
& (-5,5,0,-5,1), (-6,4,2,-5,1), (4,-4,0,-5,-1), (3,-5,2,-5,-1), \\
& (0,4,-4,-5,-1), (-2,5,-3,-5,-1), (3,1,-4,-5,1), (4,-1,-3,-5,1), \\
& (-6,0,6,-6,1), (-1,-5,6,-6,-1), (-6,5,1,-6,-1), (4,-5,1,-6,1), \\
& (-1,5,-4,-6,1), (4,0,-4,-6,-1), (-4,-4,8,-9,-1), (-5,-3,8,-9,1), \\
& (-8,4,4,-9,-1), (-8,3,5,-9,1), (3,-7,4,-9,1), (2,-7,5,-9,-1), \\
& (-4,7,-3,-9,1), (-5,7,-2,-9,-1), (6,-3,-3,-9,-1), (6,-4,-2,-9,1), \\
& (3,3,-6,-9,-1), (2,4,-6,-9,1), \ldots \}.
\end{aligned}$$

The $\pi_{\mathfrak{a}}$ projection leads to the following set of vectors:



In the figure the grade values increase to the right. The vertical line unit is the basic root vector β of $A_2^{(2)}$.

For the subalgebra $A_2^{(2)}$ the basic roots are

$$\beta = (1,0,0)\,; \quad \beta_0 = \delta - 2\beta = (-2,0,1)\,;$$

with

$$\theta = 2\beta$$

and the normalization

$$|\beta|^2 = 1, \quad |\beta_0|^2 = 4, \quad (\beta_0, \beta) = -2. \tag{37}$$

The fundamental weights

$$\omega_1 = 1/2\beta + k = (1/2, 1, 0)\,, \quad \omega_0 = 2k = (0,2,0)$$

and the Weyl vector

$$\rho = 1/2\beta + 3k = (1/2, 3, 0).$$

The positive roots are

$$\Delta^+_{A_2^{(2)}} = \left\{ \begin{array}{cc} \beta + n\delta, \pm 2\beta + (2n+1)\,\delta; & n \in \mathbf{Z}_{\geq 0} \\ -\beta + m\delta; \quad m\delta & m \in \mathbf{Z}_{>0} \end{array} \right\}$$

and have the multiplicity one. The Weyl group $W_{A_2^{(2)}}$ is generated by the classical reflection s_β and the translations $t \in T_{A_2^{(2)}} \subset W_{A_2^{(2)}}$ along the coroot $\alpha_0^\vee = 1/2\delta - \beta = (-1,0,1/2)$: $T_{A_2^{(2)}} = \left\{ t_{l\alpha_0^\vee}, l \in \mathbf{Z} \right\}$.

The injection $A_2^{(2)} \longrightarrow A_2^{(1)}$ is governed by its classical part – the special injection $B_1 \longrightarrow A_2$. The latter means that when we construct the subset $\Delta_{A_2^{(2)}}$ in the root space of $A_2^{(1)}$ the roots in $\Delta_{A_2^{(2)}}$ are scaled:

$$\beta = \alpha_1/2, \qquad K_{A_2^{(2)}} = 2K_{A_2^{(1)}}. \tag{38}$$

(So in the modules L^μ of the level k the $A_2^{(2)}$-submodules have the level $2k$.)

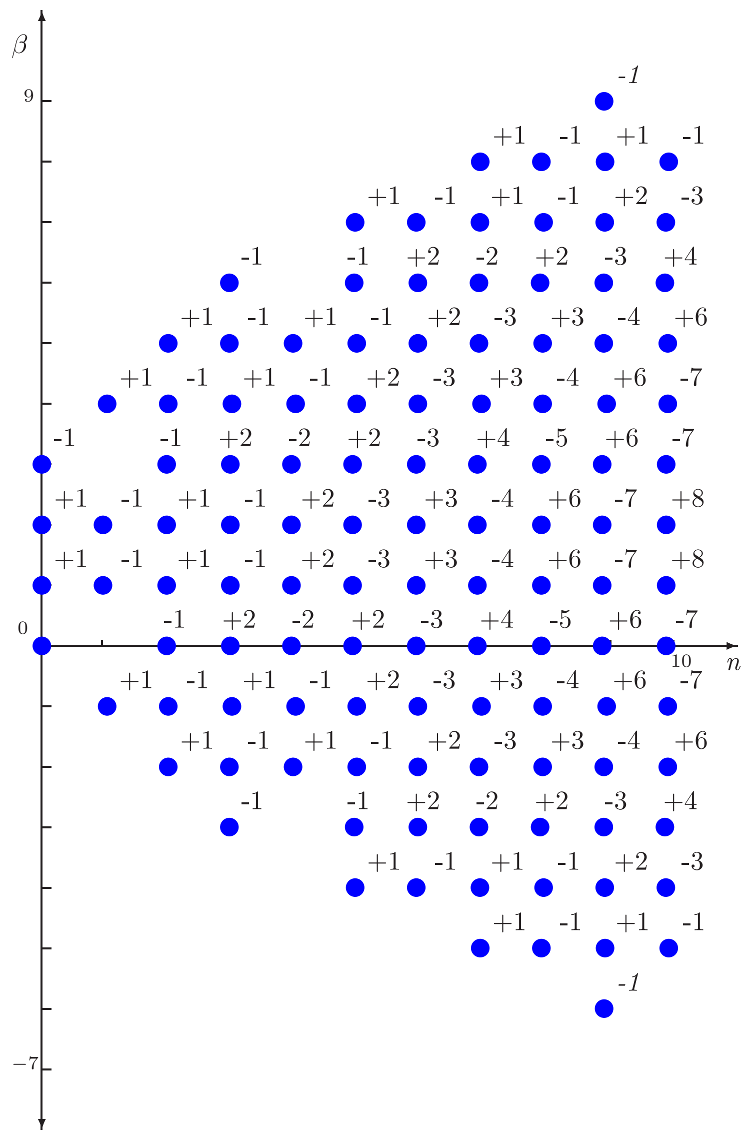
According to (13) the set $\Phi_{A_2^{(2)} \subset A_2^{(1)}}$ is defined by the opposite vectors in the nonzero components of the element $\prod_{\alpha \in \left(\pi_{A_2^{(2)}} \circ \Delta^+ \right)} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}(\alpha) - \mathrm{mult}_{\mathfrak{a}}(\alpha)}$. Taking into account the scaling (38) we obtain the element

$$\left(1 - e^{-\beta}\right)\left(1 - e^{2\beta}\right) \times \prod_{\varkappa = \pm 1} \prod_{n=1}^{\infty} \left(1 - e^{\varkappa 2\beta + 2n\delta}\right)\left(1 - e^{\varkappa\beta + n\delta}\right) \prod_{m=1}^{\infty} \left(1 - e^{+m\delta}\right)$$

generated by the set $\Phi_{A_2^{(2)} \subset A_2^{(1)}}$. Here the lowest vector γ_0 is $-\beta$ and the fan is

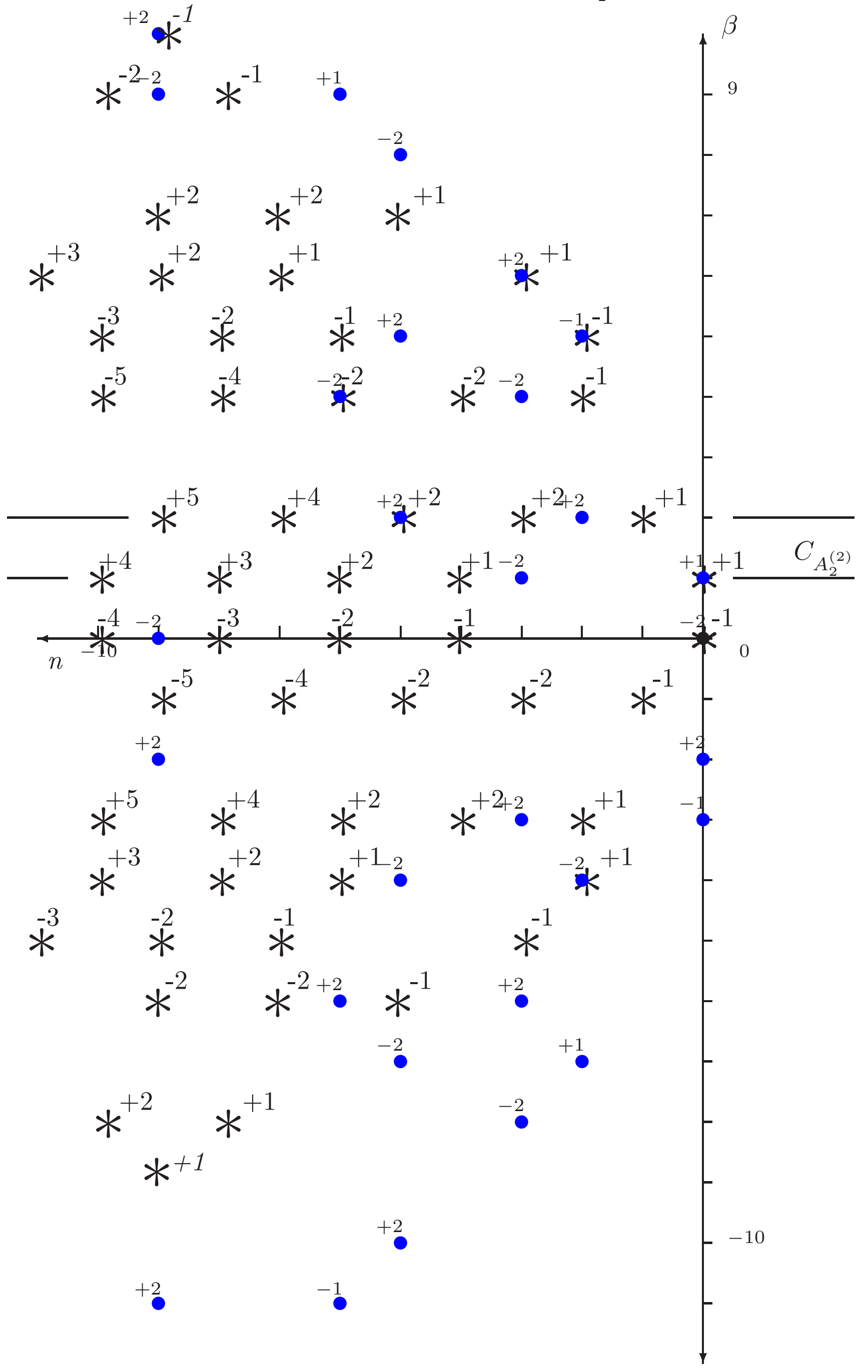
$$\Gamma_{A_2^{(2)} \subset A_2^{(1)}} = \left\{ \xi + \beta | \xi \in \Phi_{A_2^{(2)} \subset A_2^{(1)}} \right\} \setminus \{0\}\,.$$

The structure of the fan can be illustrated by the following figure presenting the vectors $\gamma \in \Gamma_{A_2^{(2)} \subset A_2^{(1)}}$ with the grade $n \leq 10$ and their multiplicities $s(\gamma)$.



Now we are able to construct the branching rules and explicitly reduce the module L^{ω_0} with respect to the subalgebra $A_2^{(2)}$. Remember that in terms of $A_2^{(2)}$ the diagram $\mathcal{N}^{\omega_0}$ is located in the subspace of level $k = 2$. Applying the formula (16) in the sublattice with $n = 0$ we find the first nontrivial value for the weight with the highest (for $n = 0$) anomalous multiplicity , that is for the

vector $(1,2,0)$ *where the branching coefficient is evident,* $k^{(\mu)}_{(1,2,0)} = 1$. *Implementing the recurrence procedure we obtain the set* $\widehat{\Psi_{A_2^{(2)}}}$ *of singular weights:*



We have performed the branching in terms of the singular weights $\widehat{\Psi^{(\xi)}_{A_2^{(2)}}}$ *of the submodules* $L^{\xi}_{A_2^{(2)}}$ *in the decomposition* $L^{\mu}_{A_2^{(1)}\downarrow A_2^{(2)}} = \bigoplus_{\xi\in P^+_{A_2^{(2)}}} b^{(\mu)}_{\xi} L^{\xi}_{A_2^{(2)}}$ *. Now it is quite easy to extract the branching coefficients* $b^{(\mu)}_{\xi}$*. The intersection*

$$\widehat{\Psi_{A_2^{(2)}}} \bigcap \overline{C}_{A_2^{(2)}}$$

gives the set of highest weights and their multiplicities $b^{(\mu)}_{\xi}$ *and the branching is*

$$\begin{aligned} L^{\omega_0}_{A_2^{(1)}\downarrow A_2^{(2)}} &= L^{\omega_0}_{A_2^{(2)}}(0) \oplus L^{2\omega_1}_{A_2^{(2)}}(-1) \oplus 2L^{2\omega_1}_{A_2^{(2)}}(-3) \oplus L^{\omega_0}_{A_2^{(2)}}(-4) \\ &\quad \oplus 2L^{2\omega_1}_{A_2^{(2)}}(-5) \oplus 2L^{\omega_0}_{A_2^{(2)}}(-6) \oplus 4L^{2\omega_1}_{A_2^{(2)}}(-7) \\ &\quad \oplus 3L^{\omega_0}_{A_2^{(2)}}(-8) \oplus 5L^{2\omega_1}_{A_2^{(2)}}(-9) \oplus 4L^{\omega_0}_{A_2^{(2)}}(-10) \oplus \dots \end{aligned}$$

(Notice that as far as we have shifted the set $\Phi_{A_2^{(2)}\subset A_2^{(1)}}$ *the Weyl chamber* $\overline{C}_{A_2^{(2)}}$ *is also shifted correspondingly.) The result can be presented in terms of two branching functions*

$$\begin{aligned} b^{(\mu)}_{I}(q) &= 1 + q^4 + 2q^6 + 3q^8 + 4q^{10} + \dots \\ b^{(\mu)}_{II}(q) &= q + 2q^3 + 2q^5 + 4q^7 + 5q^9 + \dots \end{aligned}$$

5 Conclusions

We have demonstrated that the decompositions of integrable highest weight modules of a simple Lie algebra (classical or affine) with respect to its reductive subalgebra obey the (infinite) set of algebraic relations. These relations originate from the properties of the singular vectors of the module $L_{\mathfrak{g}}$ considered as the highest weights of the Verma modules $M_{\mathfrak{a}}$. This gives rise to the recursion relations for the branching coefficients.

The properties stated above are encoded in the subset $\Gamma_{\mathfrak{g}\supset\mathfrak{a}}$ of the weight lattice $P_{\mathfrak{a}}$ called the fan of the injection. The fan depends only on the map $\mathfrak{a} \longrightarrow \mathfrak{g}$. It describes the injection (whenever it is regular or special) just as the root system describes the injection $\mathfrak{h}(\mathfrak{g}) \longrightarrow \mathfrak{g}$ of a Cartan subalgebra. Thus in the simplest case, when $\mathfrak{a} = \mathfrak{h}(\mathfrak{g})$, the recursion procedure produces the weight diagram of a module $L_{\mathfrak{g}}$.

When applied to a reduction of highest weight modules the recursion described by the fan provides a highly effective tool to obtain the explicit values of branching coefficients.

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